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ON PERSISTENT HOMEOMORPHISMS

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In this note we prove that a solenoidal group automorphism is persistent if and only if topologically stable.

§ 0. Introduction.

In [3] Lewowicz introduced the notion of persistency for a homeomorphism of a compact connected Riemannian manifold. Then he showed that every pseudo-Anosov map is persistent and by using this notion, that is structurally stable under some conditions.

In this note we define as in [3] a persistency for a homeomorphism of a compact metric space, and study a topological property of a persistent homeomorphism.

The following is proved.

Theorem. Let X be a solenoidal group, and let $\sigma : X \to X$ be a group automorphism. Then the following (1) and (2) are equivalent;

(1) (X, σ) is persistent,

(2) (X, σ) is topologically stable.

In [1] Aoki proved that (X, σ) is topologically stable if and only if (X, σ) has the pseudo-orbit tracing property. Further, there exist solenoidal automorphisms with the pseudo-orbit tracing property such that one of the following conditions holds:

(a) (X, σ) is not expansive,

(b) (X, σ) is not densely periodic.

Since every finite-dimensional torus is a solenoidal group, we have the following corollary.

Corollary. Let T^r be the r-dimensional torus, and let σ be a group automorphism of T^r. Then the following conditions are mutually equivalent;

(i) σ is persistent,

(ii) σ is topologically stable,

(iii) σ has the pseudo-orbit tracing property,

(iv) σ is expansive,

(v) σ is hyperbolic,

(vi) σ is structurally stable.

The statement is true for a group automorphism of $\mathbb{R}^r$, where $\mathbb{R}^r$ is the r-dimensional vector space (cf. [4]).

§ 1. Definitions and Examples.

Let $f : X \to X$ be a homeomorphism of a compact metric space (X, d). We denote by $\mathscr{H}(X)$ the set of all homeomorphisms of X with metric $d(f, g) = \max\{d(f(x), g(x)) : x \in X\}$ $(f, g \in \mathscr{H}(X))$. We say that an f-invariant subset $K \subset X$ is <u>persistent</u> if for each $\varepsilon > 0$ there is $\delta > 0$ with the property that for every $g \in \mathscr{H}(X)$ with $d(f, g) < \delta$ and for every $x \in K$, there is $y \in X$ such that $d(f^n(x), g^n(y)) < \varepsilon$ for every $n \in \mathbb{Z}$. When $K = X$ we say that f is <u>persistent</u>. We remark that this notion is independent of the metric for X. We call f to be <u>topologically stable</u> if for each $\varepsilon > 0$ there is $\delta > 0$ with the property that for every $g \in \mathscr{H}(X)$ with $d(f, g) < \delta$ there is a continuous map $h : X \to X$ such that $f \circ h = h \circ g$ and $d(h, id) < \varepsilon$. If X is a compact manifold and $\varepsilon > 0$ is small enough, then $d(h, id) < \varepsilon$ implies that h maps X onto itself. Therefore it is easy to see that every topologically stable homeomorphism of a compact manifold is persistent. In general case there is an example that is not true.

<u>Example 1</u>. The finite set $X_i = \{0, 1\}$ is fixed with the discrete topology for $i \in \mathbb{Z}$. Consider $X = \Pi_{i=-\infty}^{\infty} X_i$, equipped with the product topology, and the shift homeomorphism $\sigma : X \to X$ defined by $(\sigma(x))_j = x_{j+1}$ for all $j \in \mathbb{Z}$. Let d be the metric on X defined by $d(x, y) = 2^{-n}$ if n is the largest natural number with $x_j = y_j$ for all $|j| < n$, and $d(x, y) = 1$ if $x_0 \neq y_0$. It is well known that σ is topologically stable. Now we show that σ is not persistent. Put $\varepsilon = 1/4$ and fix any $\delta > 0$. Then there is $n > 0$ such that $1/2^n < \delta$. Define $g \in \mathscr{H}(X)$ by $(g(x))_j = x_j$ if $j < -n$

or $j > n$, $(g(x))_j = x_{j+1}$ if $-n \leqq j < n$, and $(g(x))_n = x_{-n}$. Obviously, $d(g, \sigma) < \delta$ and $g^{2n+1}(y) = y$ for all $y \in X$. Consider

$$x' = (\cdots, 0, 0, 1, 0, 0, 1, 0, \overset{\downarrow}{1}, 0, 1, 0, 0, 1, 0, 0, \cdots) \in X.$$

Then for all $y \in X$ with $d(x', y) < \varepsilon$, it is easy to see that $d(\sigma^{2n+1}(x'), g^{2n+1}(y)) \geqq \varepsilon$. Therefore σ is not persistent.

Let (X, d) and f be as above. Given $\delta > 0$, a sequence $\{x_j\}_{j=a}^{b}$ $(-\infty \leqq a < b \leqq \infty)$ is called a δ-<u>pseudo-orbit</u> of f if $d(f(x_j), x_{j+1}) < \delta$ for $a \leqq j \leqq b-1$. Given $\varepsilon > 0$, a sequence $\{x_j\}_{j=a}^{b}$ is said to be ε-<u>traced</u> by a point y in X if $d(f^j(y), x_j) < \varepsilon$ for $a \leqq j \leqq b$. We say that f has the <u>pseudo-orbit tracing property</u> (POTP) if for each $\varepsilon > 0$ there is $\delta > 0$ such that every δ-pseudo-orbit of f can be ε-traced by some point in X.

We say that X is <u>solenoidal</u> if X is a compact connected finite-dimensional metric abelian group.

Finally, we give two examples of persistent homeomorphisms of compact totally disconnected metric spaces.

<u>Example 2</u>. Let X be the Cantor set in $[0, 1]$: i. e. X is the set of the numbers $x \in [0, 1]$ with $x = 3^{-1}a_1 + 3^{-2}a_2 + \cdots$ ($a_i = 0$ or 2 for $i \geqq 1$). For $r \geqq 1$, we call the set $X \cap [3^{-r}i, 3^{-r}(i+1)]$ $(0 \leqq i \leqq 3^r - 1)$ a <u>Cantor subinterval with rank</u> r if $X \cap (3^{-r}i, 3^{-r}(i+1)) \neq \phi$ (see [5]). We denote by $I(i, r)$ $(i = 1, 2, 3, \cdots, 2^r)$, the i-th Cantor subinterval with rank r from the left. We show that if $f \in \mathscr{H}(X)$ is an isometry, then f is

persistent. To do this, for any $\varepsilon > 0$, fix $r > 0$ with $3^{-r} < \varepsilon$. Choose $0 < \delta < 3^{-r}$ such that if $d(f, g) < \delta$ $(g \in \mathscr{H}(X))$, then $d(f^{-1}, g^{-1}) < 3^{-r}$. For every $x \in X$ and every $j \in \mathbb{Z}$, define $i_j \in \{1, 2, 3, \cdots, 2^r\}$ by $f^j(x) \in I(i_j, r)$. Obviously, $g(x) \in I(i_1, r)$. Since f is an isometry, $d(f^2(x), fg(x)) < 3^{-r}$ and so $fg(x) \in I(i_2, r)$. On the other hand, we have that $d(fg(x), g^2(x)) < 3^{-r}$ (since $d(f, g) < \delta$), and so $g^2(x) \in I(i_2, r)$: i. e. $d(f^2(x), g^2(x)) < 3^{-r} < \varepsilon$. Continuing in this fashion, we can see that $d(f^n(x), g^n(x)) < \varepsilon$ for all $n \geqq 0$. A similar way shows that $d(f^n(x), g^n(x)) < \varepsilon$ for all $n \leqq 0$. Thus f is persistent.

<u>Example 3</u>. Let (X, d) be a compact totally disconnected metric group, and let $\sigma : X \to X$ be a group automorphism. The group operation is written by multiplicative form. We show that if (X, σ) has zero-topological entropy, then (X, σ) is persistent. It is known that every group automorphism of X has the POTP (see Application 2 of [2]). Since (X, σ) has zero-topological entropy, X contains a sequence $X = X_0 \supset X_1 \supset X_2 \supset \cdots$ of completely σ-invariant normal subgroups such that $\bigcap X_n$ is trivial and for every $n \geqq 0$, X/X_k is a finite group (cf. Lemma 14 of [2]). For each $\varepsilon > 0$, there is $k > 0$ such that $\mathrm{diam}(X_k) < \varepsilon/2$. Since X/X_k is finite, there is an integer $\ell_k > 0$ such that $X = \bigcup_{i=1}^{\ell_k} h_i X_k$ $(h_i \in X)$ and $h_i X_k \cap h_j X_k = \phi$ for $1 \leqq i \neq j \leqq \ell_k$. Thus we have that $d(h_i X_k, h_j X_k) = \inf\{d(a, b) : a \in h_i X_k, b \in h_j X_k\} > 0$ if $1 \leqq i \neq j \leqq \ell_k$ (since each $h_i X_k$ is open and closed in X). Let us put $\delta_k = \min\{\varepsilon/2, \min\{d(h_i X_k, h_j X_k) : 1 \leqq i \neq j \leqq \ell_k\}\}$. Choose $\delta = \delta(\delta_k) > 0$

as in the definition of the POTP of σ and fix $f \in \mathscr{H}(X)$ with $d(\sigma, f) < \delta$. Then for every $x \in X$, $\{f^n(x)\}_{n=-\infty}^{\infty}$ is a δ-pseudo-orbit of σ. Since σ has the POTP, there is a point $y \in X$ such that $d(\sigma^n(y), f^n(x)) < \delta_k$ for $n \in \mathbb{Z}$. Putting $n = 0$ gives $d(x, y) < \delta_k$ and so $xy^{-1} \in X_k$ (the metric d is translation invariant). Hence, we get that $d(\sigma^n(x), \sigma^n(y)) < \varepsilon/2$ for $n \in \mathbb{Z}$ since $\sigma(X_k) = X_k$. Therefore we have that

$$d(f^n(x), \sigma^n(x)) \leq d(f^n(x), \sigma^n(y)) + d(\sigma^n(y), \sigma^n(x)) < \varepsilon$$

for all $n \in \mathbb{Z}$, and so $\sigma : X \to X$ is persistent.

§ 2. Proof of Theorem.

Hereafter X is an r-dimensional solenoidal group with the invariant metric d and σ is a group automorphism of X. We write the group operation by additive form. First of all we prepare lemmas that we need. The following lemmas 1 and 2 are known (see § 1, [1]).

Lemma 1. There exist the r-dimensional vector space $\mathbb{R}^r$, a group automorphism $\gamma : \mathbb{R}^r \to \mathbb{R}^r$, a group homomorphism $\psi : \mathbb{R}^r \to X$ and a totally disconnected subgroup of X such that

(i) $\psi \circ \gamma = \sigma \circ \psi$,

(ii) $X = \psi(\mathbb{R}^r) + F$ and $\overline{\psi(\mathbb{R}^r)} = X$,

(iii) $\psi^{-1}\{\psi(\mathbb{R}^r) \cap F\} = \mathbb{Z}^r$,

(iv) there is a closed neighbourhood U of 0 in $\mathbb{R}^r$ so that $\psi : U \to X$ is an into homeomorphism, $\psi(U) \cap F = \{0\}$ and $\psi(U) + F$ is a closed neighbourhood of 0 in X (we shall write $\psi(U) \oplus F$ such a neighbourhood $\psi(U) + F$).

We call $(\mathbb{R}^r, \gamma)$ the <u>lifting system</u> of (X, σ).

<u>Lemma 2</u>. Let F be as in Lemma 1. Then F contains subgroups F^+, F^- and H such that

(i) $\sigma(H) = H$,

(ii) $F^+ \supset \sigma F^+ \supset \cdots \supset \bigcap_{n=0}^{\infty} \sigma^n(F^+) = \{0\}$,

(iii) $F^- \supset \sigma^{-1} F^- \supset \cdots \supset \bigcap_{n=0}^{\infty} \sigma^{-n}(F^-) = \{0\}$,

(iv) $\sigma F^- / F^-$ and $F^+ / \sigma F^+$ are finite,

(v) $F = F^- \oplus F^+ \oplus H$.

The following lemma is well known.

<u>Lemma 3</u>. Let $h : \mathbb{R}^r \to \mathbb{R}^r$ be a continuous map, and let $\varepsilon > 0$ be any real number. If $\|h(v) - v\|_{\mathbb{R}^r} < \varepsilon$ for all $v \in \mathbb{R}^r$, then h is a surjection. Here $\|\cdot\|_{\mathbb{R}^r}$ denotes a usual norm of $\mathbb{R}^r$.

<u>Proof</u>. Assuming that $\mathbb{R}^r \setminus h(\mathbb{R}^r) \neq \phi$, we derive a contradiction. If we take $u \in \mathbb{R}^r \setminus h(\mathbb{R}^r)$, then $u \notin h(\mathbb{R}^r)$. Hence we may assume that $0 \notin h(\mathbb{R}^r)$. For, put $h'(v) = h(v+u) - u$ for $v \in \mathbb{R}^r$. Then $h' : \mathbb{R}^r \to \mathbb{R}^r$ is a continuous map such that $0 \notin h'(\mathbb{R}^r)$ and $\|h'(v) - v\|_{\mathbb{R}^r} < \varepsilon$ for $v \in \mathbb{R}^r$. Let $H_t(v) = (1-t)v + th(v)$ for $0 \leq t \leq 1$ and $v \in \mathbb{R}^r$. Then $H_t : \mathbb{R}^r \to \mathbb{R}^r$ is a homotopy from h to $\mathrm{id}_{\mathbb{R}^r}$. Define

$$F_t^{(m)}(v) = H_t(mv) / \|H_t(mv)\|_{\mathbb{R}^r}$$

for $m > 0$, $0 \leqq t \leqq 1$ and $v \in \mathbb{R}^r$ with $H_t(mv) \neq 0$, then for a sufficiently large $m' > 0$, $F_t^{(m')} : S^{r-1} \to S^{r-1}$ $(0 \leqq t \leqq 1)$ is a homotopy from $F_1^{(m')}$ to $\mathrm{id}_{S^{r-1}}$ (since $\|h(v) - v\|_{\mathbb{R}^r} < \varepsilon$ for $v \in \mathbb{R}^r$). Since degree is homotopy invariant, we have that $\deg(F_1^{(m')}) = 1$. On the other hand, since $h(0) \neq 0$, if we choose $m'' > 0$ small enough, then $F_1^{(m'')}(S^{r-1}) \subsetneqq S^{r-1}$ and so $\deg(F_1^{(m'')}) = 0$. This is contradictory to the fact that $F_1^{(m')}$ is homotopic to $F_1^{(m'')}$.

Now we give a proof of Theorem. It was showed in [1] that (X, σ) is topologically stable if and only if the lifting system $(\mathbb{R}^r, \gamma)$ of (X, σ) is hyperbolic (see Theorems 1 and 2 of [1]). Hence, to see that (1) $\to$ (2), assuming that $(\mathbb{R}^r, \gamma)$ is not hyperbolic, we prove that (X, σ) is not persistent.

As usual $\mathbb{R}^r = E^s \oplus E^c \oplus E^u$ where E^s, E^c and E^u are the subspaces corresponding to the eigenvalues of γ with modulus less than one, equal to one and greater than one respectively. Let $|\cdot|_s$ and $|\cdot|_u$ be some norms on E^s and E^u respectively. Since $E^c \neq \{0\}$, by using Jordan's normal form in the real field for (E^c, γ), we get a finite direct sum $E^c = E^{c0} \oplus \cdots \oplus E^{ck}$ of the subspaces E^{ci} satisfying the following conditions; for $0 \leqq i \leqq k$, the dimension of E^{ci} is 1 or 2, and

$$\gamma_{E^c} = \begin{pmatrix} \gamma_0 & I_1 & & 0 \\ & \gamma_1 & \ddots & \\ & & \ddots & I_k \\ 0 & & & \gamma_k \end{pmatrix}$$

where $\gamma_i : E^{ci} \to E^{ci}$ is an isometry under some norm $|\cdot|_{c_i}$ of E^{ci}

and each $I_i : E^{c_i} \to E^{c_{i-1}}$ is either a zero map or a map corresponding to the identity matrix. Define a norm $|\cdot|_c$ of E^c by

$$|v|_c = \max\{|v^i|_{c_i} : 0 \leqq i \leqq k\} \quad (v = v^0 + \cdots + v^k \in \oplus_{i=0}^{k} E^{c_i}).$$

Clearly

$$\|v\| = \max\{|v^s|_s, |v^c|_c, |v^u|_u\} \quad (v = v^s + v^c + v^u \in \mathbb{R}^r)$$

is equivalent to the usual norm of $\mathbb{R}^r$. If $B(\alpha) = \{v \in \mathbb{R}^r : \|v\| \leqq \alpha\}$ for $\alpha > 0$, then there is $\alpha_1 > 0$ such that $\psi(B(\alpha_1)) \oplus F$ is a closed neighbourhood of 0 in X (by Lemma 1 (iv)). For $x = x_1 + x_2$ with $x_1 \in \psi(B(\alpha_1))$ and $x_2 \in F$, put

$$\rho(x) = \max\{\alpha_1, \max\{\|\psi^{-1}(x_1)\|, d(x_2, 0)\}\}$$

and define a metric d_1 for X by

$$d_1(x, y) = \begin{cases} \rho(x, y) & \text{if } x - y \in \psi(B(\alpha_1)) \oplus F \\ \alpha_1 & \text{otherwise.} \end{cases}$$

The metric d_1 is compatible with the original topology of X and in particular $d_1(\psi(v), 0) = \|v\|$ for $v \in B(\alpha_1)$. For $\alpha \in (0, \alpha_1)$, we define $F(\alpha) = \{x \in F : d_1(x, 0) \leqq \alpha\}$. Since

$$F' = \bigcap_{n=-1}^{1} \sigma^n(F^+) \oplus \bigcap_{n=-1}^{1} \sigma^n(F^-) \oplus H$$

is an open subgroup of F (by Lemma 2), there is $\beta > 0$ $(\beta < \alpha_1/2)$ such that $F(\beta) \subset F'$. Here we may assume that the number β is chosen so that $B(\beta) \subset \bigcap_{n=-1}^{1} \gamma^n(B(\alpha_1))$. Put $E = E^{c_0}$ and $E' = E^{c_1} \oplus \cdots \oplus E^{c_k} \oplus E^s \oplus E^u$. For any $v \in \mathbb{R}^r = E \oplus E'$, let $v = (v_1, v_2, \cdot$

$\cdots, v_r)$ be the representation by components with respect to the fundamental vector of $\mathbb{R}^r = E \oplus E'$. Put $\varepsilon = \beta/8$ and fix any $\delta > 0$ $(\delta < \varepsilon)$. Let $\phi : \mathbb{R}^r \to \mathbb{R}^r$ be the time-one map for the vector field (*) given by

$$(*) \qquad \dot{v}_i = \delta' \chi(v_1) \cdots \chi(v_r) v_i \quad \text{for} \quad 1 \leqq i \leqq r,$$

where $\chi : \mathbb{R} \to \mathbb{R}$ is a function of class C^∞ such that $0 < \chi(t) < 1$ $(\beta/2 < |t| < 2\beta/3)$, $\chi(t) = 1$ $(|t| < \beta/2)$ and $\chi(t) = 0$ $(2\beta/3 \leqq |t|)$, and $\delta' > 0$ is a real number chosen such that $\|\phi(v) - v\| < \delta$ for $v \in \mathbb{R}^r$. Let $\tilde{\phi}$ be a map from $\psi(B(\alpha_1)) \oplus F$ onto itself defined by

$$\tilde{\phi}(x) = \begin{cases} \psi(v) + f & \text{if} \quad f \notin F' \\ \psi(\phi(v)) + f & \text{if} \quad f \in F' \end{cases}$$

for $x = \psi(v) + f \in \psi(B(\alpha_1)) \oplus F$. We shall denote again by $\tilde{\phi}$ the extension on X as $\tilde{\phi}(x) = x$ for $x \notin \psi(B(\alpha_1)) \oplus F$. Define a map $g : X \to X$ by $g(x) = \tilde{\phi} \circ \sigma(x)$ $(x \in X)$. Obviously, $d_1(\sigma, g) < \delta$ and $g \in \mathscr{H}(X)$. Consider $x' = \psi(u)$ where $u = (\beta/4, 0, 0, \cdots, 0) \in E \oplus E' = \mathbb{R}^r$. Then we get

$$d_1(\sigma^n(x'), 0) = d_1(\psi(\gamma^n(u)), 0) = \|\gamma^n(u)\| = \beta/4$$

for all $n \geqq 0$. For any

$$y \in W_\varepsilon(x') = \{z \in X : d_1(z, x') \leqq \varepsilon\} = \psi(B(\varepsilon)) \oplus F(\varepsilon) + x',$$

there are $w \in B(\varepsilon)$ and $f \in F(\varepsilon)$ such that $y = \psi(w + u) + f$. It is clear that $\beta/8 < \|\pi_E(w + u)\| < 3\beta/8$, where $\pi_E : \mathbb{R}^r \to E$ denotes a projection along complementary subspace E'. Hence there is the smallest integer $n_0 \geqq 0$ such that $3\beta/8 < \|(\phi\gamma)^{n_0}(w + u)\| < \alpha_1$ or $d_1(\sigma^{n_0}(f), 0) > 3\beta/8$ $(\sigma^{n_0}(f) \in F)$ holds. Since $\psi_{B(\alpha_1)}$ is an isometry, we can easily obtain that $d_1(g^{n_0}(y), 0) > 3\beta/8$, and so $d_1(\sigma^{n_0}(x'), g^{n_0}(y)) > \beta/8 = \varepsilon$. Therefore (X, σ) is not persistent.

To see that $(2) \rightarrow (1)$, we show that if $(\mathbb{R}^r, \gamma)$ is hyperbolic, then (X, σ) is topologically stable and a continuous map $h : X \rightarrow X$ is onto. To get the conclusion, it is enough to check that a continuous map h constructed in the proof of the statement $(B) \rightarrow (A)$ of [1] (see pp. 133–135 and Correction) is onto. This is sketched as follows (see [1] for details).

There is a 1-to-1 group homomorphism $\psi^* : \mathbb{R}^r/_{\mathrm{Ker}\,\psi} \rightarrow \psi(\mathbb{R}^r)$. In [1], $\mathbb{R}^r/_{\mathrm{Ker}\,\psi}$ is denoted by the symbol $V_1 \oplus V_2$. Remark that $\mathrm{Ker}\,\psi \subset \mathbb{Z}^r$ by Lemma 1 (iii). Let $\tilde{d}_0$ denote the metric induced on $V_1 \oplus V_2$ by the metric d_0 of $\mathbb{R}^r$. We note that d_0 is equivalent to the Euclidean metric on $\mathbb{R}^r$ (see [1, p. 123]). Let $\tilde{\gamma} : V_1 \oplus V_2 \rightarrow V_1 \oplus V_2$ denote the map induced by γ. Obviously, $\psi^* \circ \tilde{\gamma} = \sigma \circ \psi^*$. Since γ is hyperbolic, $\tilde{\gamma}$ is topologically stable (see [1, pp. 131–132] or [4]). For any $\varepsilon > 0$ (very small), let $\delta > 0$ be the number with the property of topological stability. Take and fix any $f \in \mathscr{H}(X)$ with $d_1(f, \sigma) < \delta$. Then there is a sequence $\{f_n\}_{n=0}^{\infty} \subset \mathscr{H}(X)$ such that $f_n(\psi(\mathbb{R}^r)) = \psi(\mathbb{R}^r)$ $(n \geqq 0)$, $d_1(f_n, \sigma) < \delta$ for n large enough and $f_n \rightarrow f$ $(n \rightarrow \infty)$. Fix an integer n such that $d_1(f_n, \sigma) < \delta$, and put $\tilde{f}_n(v) = \psi^{*-1} \circ f_n \circ \psi^*(v)$ for $v \in V_1 \oplus V_2$. Then $\tilde{f}_n : V_1 \oplus V_2 \rightarrow V_1 \oplus V_2$ is a homeomorphism and $\tilde{d}_0(\tilde{f}_n(v), \tilde{\gamma}(v)) < \delta$ for $v \in V_1 \oplus V_2$. So there is a continuous map $\tilde{h}_n : V_1 \oplus V_2 \rightarrow V_1 \oplus V_2$ such that $\tilde{h}_n \circ \tilde{f}_n = \tilde{\gamma} \circ \tilde{h}_n$ and $\tilde{d}_0(\tilde{h}_n(v), v) < \varepsilon$ $(v \in V_1 \oplus V_2)$. Since the natural projection $p : \mathbb{R}^r \to V_1 \oplus V_2$ is a covering projection, there is a lifting $\bar{h}_n : \mathbb{R}^r \to \mathbb{R}^r$ of $\tilde{h}_n$ such that $d_0(\bar{h}_n(v), v) < \varepsilon$ for $v \in \mathbb{R}^r$. Hence by Lemma 3, $\bar{h}_n$ maps $\mathbb{R}^r$ onto itself, and so $\tilde{h}_n(V_1 \oplus V_2) = V_1 \oplus V_2$ (since $\tilde{h}_n \circ p = p \circ \bar{h}_n$). Put $h_n = \psi^* \circ \tilde{h}_n \circ \psi^{*-1}$. Then for an arbitrarily large n, we get that $h_n \circ f_n = \sigma \circ h_n$ on $\psi(\mathbb{R}^r)$, $d_1(h_n(x), x) < \varepsilon$ $(x \in \psi(\mathbb{R}^r))$, $h_n(\psi(\mathbb{R}^r)) = \psi(\mathbb{R}^r)$, and h_n is uniformly continuous (see [1, Correction]). Thus, h_n is extended to a surjective continuous map of X since $\overline{\psi(\mathbb{R}^r)} = X$ by Lemma 1 (ii). We shall denote it by the same symbol. Since $\{h_n\}$ converges uniformly to some continuous map h of X (see [1, Correction]), it follows that $h \circ f = \sigma \circ h$ on X, $d_1(h, id) \leqq \varepsilon$ and $h(X) = X$. The proof is complete.

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